Self-organized growth model for the quenched Herring-Mullin equation

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We introduce a simple self-organized growth model mimicking the dynamics of a driven tensionless interface in a random medium near the depinning threshold. The roughness and growth exponents for the model are obtained as $\zeta \simeq 1.93$ and $\beta \simeq 0.96$, respectively. We discuss the possible continuum equation describing the motion of a driven interface in our model.

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Recently there have been many studies about the problem of a driven interface in a random medium because it is related to many physical phenomena such as fluid invasion in porous media [1,2], depinning of charge density waves [3], fluid imbibition in paper [4], driven flux motion in a type-II superconductor [5,6], etc. The motion of a driven interface in a random medium (DIRM) is determined by interplay between the external driving force and the quenched disorder in a random medium. The velocity of a driven interface is zero when the driving force F is smaller than the pinning strength induced by the quenched disorder. There exists a threshold of the driving force F_c above which the interface moves with a constant velocity. Accordingly, the velocity is zero for $F < F_c$, and it increases for $F > F_c$. This phenomenon is called the pinning-depinning transition.

The dynamics of the DIRM can be explained by the Langevin-type continuum equation. The simplest well-known Langevin-type continuum equation describing the motion of the DIRM is the quenched Edwards-Wilkinson (QEW) equation [7],

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) + F + \eta(x,h), \tag{1}$$

where $h(x,t)$ means the height of the interface at position x and time t. $\nu \nabla^2 h(x,t)$ describes the smoothening effect of interface tension. F is an external driving force, and $\eta(x,h)$ is a quenched noise with $\langle \eta(x,h) \rangle = 0$ and $\langle \eta(x,h)\eta(x',h') \rangle = 2D\delta(x-x')\Delta(h-h')$. $\Delta(z)$ is a monotonically decreasing function of z for $z>0$ and decays exponentially to zero over a finite distance a. The quenched noise term describes a random force by quenched disorder. An interesting feature of the growing interface is nontrivial scaling behavior in the global interface width. The global interface width, defined by $W(L,t) = \langle L^{-d} \Sigma_i [h_i(t) - \overline{h}(t)]^2 \rangle^{1/2}$, scales as

$$W(L,t) \sim \begin{cases} t^{\zeta/z} & \text{if} \quad t \ll L^z \\ L^{\zeta} & \text{if} \quad t \gg L^z. \end{cases} \tag{2}$$

Here $\overline{h}$, L, d, and $h_i(t)$ denote the mean height, the system size, the substrate dimension, and the height at time t and site i, respectively. ζ, z, and $\beta (= \zeta/z)$ are called the roughness, the dynamic, and the growth exponent, respectively.

Many analytical and numerical studies have been carried out to describe and understand the driven motion of an interface in a random medium [8–14]. In spite of many of these efforts, only several stochastic growth models mimicking the motion of the DIRM have been introduced. The models, for example, are the Sneppen model [15], the directed percolation depinning model [4], the random field Ising model [16], etc. Recently we introduced the model that mimics the dynamics of the DIRM with interface tension near the threshold [17]. The scaling properties of our model are in good agreement with those expected from the analytical and numerical studies of the QEW equation. In this paper, we introduce a simple stochastic growth model for the dynamics of the DIRM with negligible interface tension near the threshold. So far, there has not been any study about dynamics of the DIRM with negligible interface tension through the stochastic growth model. Hence it would be very interesting to introduce a simple stochastic growth model for the DIRM with negligible interface tension and discuss a possible continuum equation describing the motion of a driven interface in the model.

Many stochastic models for dynamics of the fluctuating interfaces in homogeneous media (FIHM) were introduced [7]. All the stochastic growth in the models occur through an interplay between the following two effects: the white noise effect of deposition of a particle at a randomly selected site on the interface, and the local relaxation effect of the deposited particle, which prevents the interface from becoming rough due to the white noise effect. As an example of the local relaxation, let us consider the Family model [18]. In the Family model, the deposited particle diffuses randomly to one of the nearest-neighbor sites whose height is lower than that of the randomly selected site. Diffusion of the deposited particle generates interface tension in the growing interface. The scaling properties obtained from the Family model are in good agreement with those expected from the Edwards-Wilkinson (EW) equation,

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) + \eta(x,t), \tag{3}$$

where $\eta(x,t)$ is a white noise with $\langle \eta(x,t) \rangle = 0$ and $\langle \eta(x,t)\eta(x',t') \rangle = 2D\delta(x-x')\delta(t-t')$. Growth of the interface in the EW equation is determined by the interplay between the interface tension term and the white noise term. It is well known that the scaling exponents obtained from the

Monte Carlo simulation of the models for the FIHM are in excellent agreement with those expected from the continuum equation describing the motion of the FIHM [7]. In the DIRM near the depinning threshold, growth of the driven interface occurs at the site having the lowest pinning strength among the whole quenched disorder on the interface. Therefore, when one designs the stochastic growth model for the DIRM near the threshold, the growth rule can be defined as follows: we preassign random numbers that represent quenched disorder in a random medium to all perimeter sites of the interface. At each time step, we add a particle on the selected site, which has a global minimum random number on the interface, and then an existing random number on that site is updated. After that, the updated particle can diffuse to its nearest-neighbor sites. When we used the diffusion rule introduced by Family in the model for the DIRM, we obtained the same scaling behavior as that of the QEW equation [17]. Another stochastic growth model for the DIRM near the threshold, which has the same relaxation rule as the restricted solid-on-solid (RSOS) model for the FIHM [19,20], was introduced by Sneppen. The scaling behaviors of the Sneppen model are the same as those of the continuum equation describing the dynamics of the interface formed from the RSOS model, which has the quenched noise term instead of the white noise term [7],

$$\frac{\partial h(x,t)}{\partial t}=\nu\nabla^2 h(x,t)+\frac{\lambda}{2}[\nabla h(x,t)]^2+\eta(x,h). \qquad (4)$$

This equation is called the quenched Kardar-Parisi-Zhang (QKPZ) equation. In the QKPZ equation, the term $\lambda/2[\nabla h(x,t)]^2$ denotes the lateral growth effect of the growing interface. This fact indicates that any other terms, which do not exist in the ontinuum equations describing dynamics of an interface formed from the model for the FIHM, are not induced from the diffusion of the newly updated particle in the model for the DIRM. It is thus possible to design a stochastic growth model for the DIRM near the threshold, which has the same dynamic rule except for the noise effect as the model for the FIHM, if one knows the stochastic growth rule of the model for the FIHM.

We now want to introduce a simple stochastic growth model for the DIRM with negligible interface tension near the threshold. It is necessary to know the stochastic growth model for the FIHM without interface tension. Several years ago, Kim and Sarma [21] introduced a stochastic model, the larger curvature (LC) model, which mimics the motion of the FIHM with negligible interface tension [21]. The exponents obtained from the model are in good agreement with those expected from the Herring-Mullin (HM) equation [22],

$$\frac{\partial h(x,t)}{\partial t}=-K\nabla^4 h(x,t)+\eta(x,t), \qquad (5)$$

where $-K\nabla^4 h(x,t)$ denotes the effect of minimizing of the local curvature of the interface. In the LC model, interface tension and lateral growth effects in a growing interface do not exist. The relaxation rule in the Kim-Sarma model is as follows: the newly added particle can diffuse randomly to one of the nearest-neighbor sites whose curvature is larger than that of the selected site. The curvature at site i is defined as $h_{i+1}-2h_i+h_{i-1}$. This relaxation tends to minimize the local curvature of the interface. Hence we can design a stochastic growth model for the DIRM without interface tension near the threshold. The growth rule in our model is as follows. (i) We preassign random numbers on the interface. (ii) We add a particle on the site having a global minimum random number on the interface, and then we change the random number on that site. (iii) The newly added particle diffuses randomly to one of the nearest neighbor sites whose curvature is larger than that of the selected site. (iv) We update the random number at the newly occupied site. The stochastic growth rule of our model is depicted in Fig. 1. In this way, we can generate a movement decreasing curvature near the minimum random number site. This is a characteristic feature appearing in the model without interface tension.

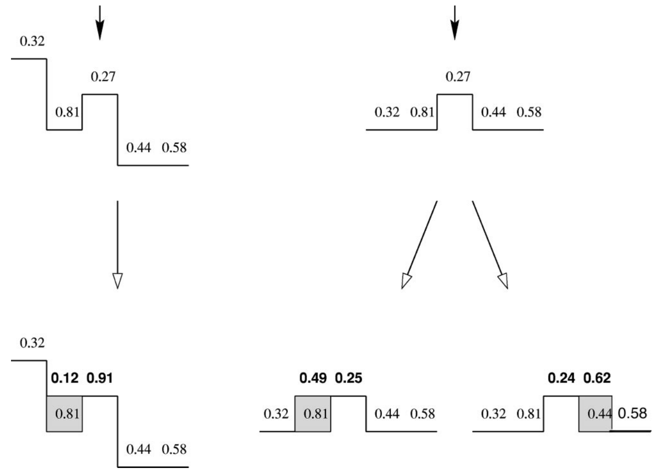

FIG. 1. Schematic representations of the stochastic rule of our model. The arrows indicate the selected sites having he lowest global minimum random number. The gray squares denote the newly added particles on the interface. The bold numbers denote the newly updated random numbers.

We have carried out Monte Carlo simulations for system sizes $L=32$, 40, 64, 90, 128, 180, 256, 362, and 512. Numerical data are averaged typically over 100 configurations. In order to obtain the growth exponent for our model, we measure the time-dependent behavior of the global interface width $W(L,t)$ starting from the initially flat interface. We plot $W^2(L,t)$ vs time in double logarithmic scale in Fig. 2. The interface width initially grows with the growth exponent, 0.5, as random growth. After that, the interface width grows with the exponent $\beta\simeq 0.96$ as shown in Fig. 2. We also consider another growth exponent by measuring the global width starting from the saturated interface instead of the flat interface. The growth exponent is measured as $\beta_s \simeq 0.79$. The exponent β_s is smaller than β obtained from the flat interface. In the case of stochastic models for the dynamics of the DIRM, the values of β_s are smaller than that of β obtained from the flat interface [15,17]. We also plot the saturated value of $W^2(L)$ vs the system size L in double logarithmic scale to obtain the roughness exponent. The obtained roughness exponent is $\zeta\simeq 1.93$ as shown in Fig. 3. We have also measured the height-height correlation function $C(x)$ defined as

$$C(x)=\left\langle \frac{1}{L^{d'}}\sum_{x}\,[h(x+x_1,\tau)-h(x_1,\tau)]^2\right\rangle^{1/2}, \qquad (6)$$

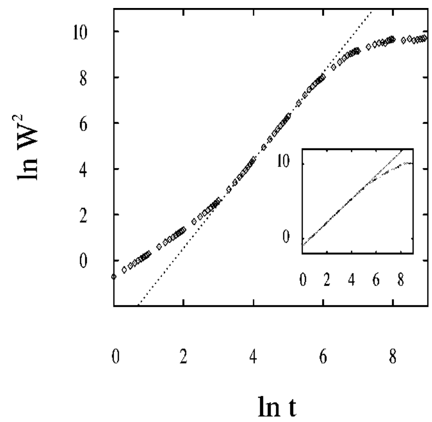

FIG. 2. Plot of $W^2(L,t)$ vs time t in double logarithmic scale for the system size $L=512$. The slope of the dotted line is $2\beta=1.92$. Inset: The same plot of $W_s^2(L,t)$ starting from the saturated interface. The slope of the line is $2\beta_s=1.58$.

where time τ is larger than the saturation time, and $C(x)$ scales as x^{ζ_c}. The roughness exponent value from $C(x)$ is $\zeta_c \simeq 0.98$ as shown in the inset of Fig. 3. This value is smaller than the one obtained from the global interface width. It is known that this anomalous scaling of the local width is due to the super-roughening, in such a way that the roughness exponent ζ_c obtained from the height-height correlation function is smaller than the one obtained from the saturated value of $W^2(L)$ [12–14]. Super-rough scaling occurs when the roughness exponent of the global width is $\zeta > 1$. Super-rough interfaces do not represent the self-affine scaling nature, since the basic step is a diverging quantity [14]. The height-height correlation function in the super-rough interfaces might be given as $C(x) \sim x^{\zeta_c} L^{\zeta-\zeta_c}$. If the correlation length x is the same as the system size L, we recover $C(x=L)=W(L)$. The value of obtained exponents is larger than that from the QEW and the QKPZ equations. In Figs. (2) and (3), the data do not show any crossover behavior. Therefore our model belongs to a different universality class from the QEW and the QKPZ universality classes.

It would be interesting to consider the continuum equation

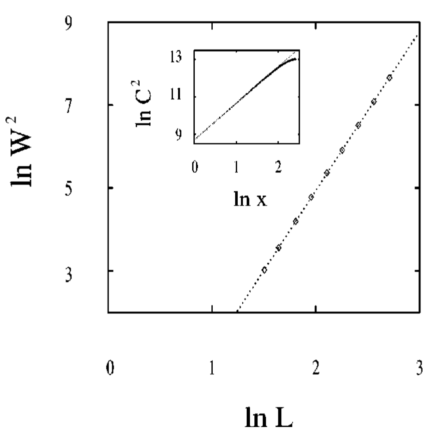

FIG. 3. Plot of $W^2(L,t)$ vs the system size L in double logarithmic scales for the system sizes $L=32$, 40, 64, 90, 128, 180, 256, 362, and 512. The slope of the line is $2\zeta=3.85$. Inset: Plot of $C^2(x)$ vs x in double logarithmic scales for the system size $L=512$. The line obtained from the least-square fits has the slope $2\zeta_c=1.96$.

describing the motion of the interface formed from our stochastic growth model. The general Langevin-type equation describing growing interface near the threshold can be written as

$$\frac{\partial h(x,t)}{\partial t} = \nu \nabla^2 h(x,t) - K\nabla^4 h(x,t) + \frac{\lambda}{2}[\nabla h(x,t)]^2 + \eta(x,h) + F_c . \quad (7)$$

A method to check whether or not there exists a surface tension effect in our model is to measure the hopping rate of the newly updated particles according to the slope of a tilted substrate after saturation [7]. When there exists a surface tension effect in the stochastic growth model, the newly updated particles tend to move in the downhill direction of the tilted substrate in the relaxation process after update. It is well known that the diffusion process of a deposited particle in the LC model is independent on the slope of the tilted substrate. In our model the random number distribution on the interface is independent on the slope of the tilted substrate. Hence, the process of selecting the site having a global minimum random number is also independent on the slope of the tilted substrate. From the above facts we believe that there is no interface tension effect in our model. Next, we check whether or not there is any lateral growth effect in the dynamics of our model. In the stochastic model with a lateral growth effect, the number of particles added on the surface generally increases as the slope of the tilted substrate becomes larger because of the presence of $\lambda/2[\nabla h(x,t)]^2 (\lambda > 0)$, so that the velocity of the driven interface depends on the slope of the tilted substrate [7]. The velocity of our model is always $v=(1/L)\{\Sigma_x^L h(x,t+1) - \Sigma_x^L h(x,t)\} = 1/L$ regardless of the slope of the substrate tilt. It is because only one particle in our model is added on the interface per each time. The stochastic growth in our model does not include any surface tension and lateral growth effects. Accordingly we propose that the continuum equation corresponding to the dynamics of our model is the quenched Herring-Mullin (QHM) equation [23],

$$\frac{\partial h(x,t)}{\partial t} = -K\nabla^4 h(x,t) + \eta(x,h). \quad (8)$$

Besides the term $-K\nabla^4 h$ in the QHM equation, there is the possibility of the existence of other terms or higher order nonlinear terms in dynamics of our model. It is impossible to check whether those effects exist or not through the simulation of the model. However, we think that no effects of those other terms exist due to the quenched noise because we could not find any crossover behavior in our model.

In summary, we have introduced a simple self-organized stochastic growth model for the driven interface in a random medium near the depinning threshold. The obtained roughness and growth exponents are $\zeta \simeq 1.93, \beta \simeq 0.96$ in $1+1$ dimensions. These are alternate exponents. Our model is designed to mimic the motion of the driven tensionless interface in a random medium near the threshold. In the model, we have used the relaxation rule introduced by Kim and Sarma [21]. The relaxation rule generates the effect of minimizing of the local curvature of the driven interface. We have thus proposed that a suitable continuum equation for dynamics of the model is the QHM equation, by showing that the interface tension and the lateral growth effects do not exist in the driven interface formed by the model.

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[23] When we were studying about the stochastic model in the QHM universality class, we came to know that Professor J.M. Kim also was studing about the QHM equation through numerical integration of the equation and automata models, which are supposed to belong to the QHM universality class.